Scattering kinematics: Transformation of differential cross sections between two moving frames[a)]

Gary L. Catchen[b)]

Department of Chemistry, Columbia University, New York, New York 10027

Javed Husain[c)] and Richard N. Zare

Department of Chemistry, Stanford University, Stanford, California 94305
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A general method is presented for transforming scattering cross sections between two moving frames. The Jacobian of the transformation is developed for nonrelativistic as well as relativistic particle energies in the case of either elastic or inelastic scattering.

I. INTRODUCTION

The primary datum of a scattering experiment is the differential cross section. It is defined as the number of particles (events) per second per solid angle observed by the detector divided by the number of particles per unit area per second incident per target particle (scattering center).[1] In many scattering experiments the detector is most conveniently described using the fixed reference frame of the laboratory, while the scattering events are most conveniently analyzed in a reference frame moving with uniform velocity with respect to the laboratory frame. In such cases it is necessary to relate the differential cross section measured in one frame to the differential cross section that would be measured in the other. This kinematic problem can be vexing, causes much confusion, and is the subject of this article.

Scattering data usually consist of either single-particle events or as coincident events. The latter is defined by two events that reach their respective detectors within some specified time delay. For single-particle data, one normally wishes to transform the laboratory scattering data to the center-of-mass reference frame. For coincidence data, again one may desire to transform to the center-of-mass frame, or to a reference frame moving with the other coincident particle, or its kinematic partner.

Figure 1 illustrates all of these types of transformations. The unprimed coordinate system (XYZ) refers to the laboratory frame, the primed coordinate system $(X'Y'Z')$ to a reference frame that moves with a speed and a direction given by the velocity vector $\mathbf{v}_B$. Thus a particle having velocity $\mathbf{v}_A$ in the $X'Y'Z'$ frame will have a velocity $\mathbf{v}_C$ in the XYZ frame, where the vector equation

$$\mathbf{v}_C = \mathbf{v}_A + \mathbf{v}_B \tag{1}$$

defines the transformation.

When the Cartesian axes of the two frames are parallel, the spherical coordinates of the three vectors are those shown in Fig. 1, i.e., the lengths are V_A, V_B, V_C, the polar angles are θ_A, θ_B, θ_C, and the azimuthal angles are ϕ_A, ϕ_B, ϕ_C.

The problem faced in transforming cross sections in one frame to another is the requirement that the transformations conserve flux. If the scattering event has no restriction on either the angle or velocity of the scattered particle, then the differential cross sections (which are proportional to the probability of a scattering event being detected with particle velocity between v and $v+dv$ and at a solid angle between Ω and $\Omega+d\Omega$ in the two frames are related by

$$\frac{d^2\sigma}{dv_A\,d\Omega_A} = J\left(\frac{v_C, \Omega_C}{v_A, \Omega_A}\right)\frac{d^2\sigma}{dv_C\,d\Omega_C}, \tag{2}$$

where

$$J\left(\frac{v_C, \Omega_C}{v_A, \Omega_A}\right) = J\left(\frac{v_C, \cos\theta_C, \phi_C}{v_A, \cos\theta_A, \phi_A}\right) \tag{3}$$

is the Jacobian of the transformation. The explicit form of the Jacobian is the 3×3 determinant:

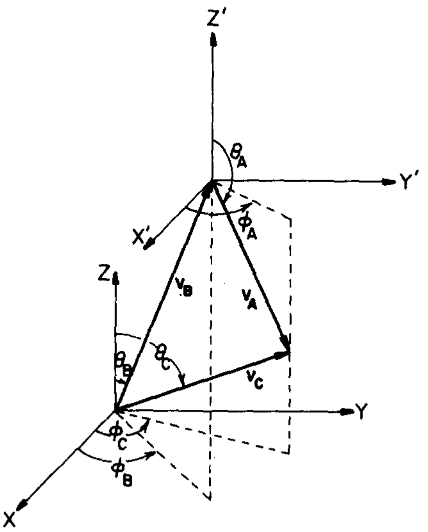

FIG. 1. The general coordinate transformation where the unprimed axes refer to the laboratory and the primed axes to the moving reference frame. The polar angles, θ_A and θ_C, range from 0 to π radians, and the azimuthal angles, ϕ_A and ϕ_C, range from 0 to 2π radians.

[a)]Dedicated to the memory of Professor J. M. Miller who inspired this work.

[b)]1976–1977 J. L. R. Morgan Predoctoral Fellow.

[c)]Graduate student, Department of Physics, Columbia University, New York, NY 10027.

$$J\left(\frac{v_C,\Omega_C}{v_A,\Omega_A}\right)=\begin{vmatrix} \frac{\partial v_C}{\partial v_A} & \frac{\partial v_C}{\partial \cos\theta_A} & \frac{\partial v_C}{\partial \phi_A} \\ \frac{\partial \cos\theta_C}{\partial v_A} & \frac{\partial \cos\theta_C}{\partial \cos\theta_A} & \frac{\partial \cos\theta_C}{\partial \phi_A} \\ \frac{\partial \phi_C}{\partial v_A} & \frac{\partial \phi_C}{\partial \cos\theta_A} & \frac{\partial \phi_C}{\partial \phi_A} \end{vmatrix} . \qquad (4)$$

In the case where restrictions on the scattering process occur, the differential cross sections must be transformed appropriately. The quantization (discretization) of exit channel velocities is such an instance. The most commonly encountered example is elastic scattering, but there are a growing number of experiments measuring state-to-state inelastic or reactive scattering where restrictions on the exit channel velocities also apply. Although the final velocity of a particle undergoing elastic scattering is constrained to one fixed value in the center-of-mass frame, the spherical polar angles, θ and ϕ, may assume a continuum of values. Therefore, the differential cross sections for elastic scattering transform according to

$$\frac{d\sigma}{d\Omega_A}=J\left(\frac{\Omega_C}{\Omega_A}\right)\frac{d\sigma}{d\Omega_C}, \qquad (5)$$

where

$$J\left(\frac{\Omega_C}{\Omega_A}\right)=J\left(\frac{\cos\theta_C,\phi_C}{\cos\theta_A,\phi_A}\right) \qquad (6)$$

is the Jacobian of the transformation, having the explicit form of the 2×2 determinant:

$$J\left(\frac{\Omega_C}{\Omega_A}\right)=\begin{vmatrix} \frac{\partial \cos\theta_C}{\partial \cos\theta_A} & \frac{\partial \cos\theta_C}{\partial \phi_A} \\ \frac{\partial \phi_C}{\partial \cos\theta_A} & \frac{\partial \phi_C}{\partial \phi_A} \end{vmatrix} . \qquad (7)$$

In the general case where many discrete exit channels j are open, the transformation becomes a sum over the j channels:

$$\sum_{i=1}^{n}\left(\frac{d\sigma(v_A,\Omega_A)}{d\Omega_A}\right)_i = J_j\left(\frac{\Omega_C(v_C,\Omega_C)}{\Omega_A(v_A,\Omega_A)}\right)\left(\frac{d\sigma(v_C,\Omega_C)}{d\Omega_C}\right)_j \qquad (8)$$

In Eq. (8) the jth term contributes one cross section in the A frame when the transformation is single valued, n terms when the transformation is multivalued. Equation (8) is just a generalization of Eq. (5). Furthermore, the preceding discussion pertains not only to discrete velocities with continuous angular distributions of the scattered products, but to the case where the velocity distribution is continuous and the angular distribution is discrete. For example, when a reaction is strongly plane polarized, the differential cross section may be treated as if it were continuous in polar angle θ, continuous or discrete in velocity, v, and discrete in azimuthal angle, ϕ. The proper Jacobian of the transformation is represented by the determinant involving the continuous variables.

When many channels are open and the detection system of the experiment is not able to resolve the discrete states, then what is inherently quantized (discrete) data may be treated as a continuous spectrum. In such circumstances, the transformation of such data between two moving frames is to be carried out using Eq. (2). In the Appendix we present in more detail the conditions under which this approach is valid.

The following discussion describes a general method for evaluating Jacobians of the frame transformation.

II. NONRELATIVISTIC TRANSFORMATION

The determination of the Jacobian involves the calculation of the partial derivatives that are shown in Eqs. (4) or (7). Equation (1) can be expressed in Cartesian components as:

$$v_C\cos\theta_C=v_B\cos\theta_B+v_A\cos\theta_A , \qquad (9)$$

$$v_C\sin\theta_C\sin\phi_C=v_B\sin\theta_B\sin\phi_B+v_A\sin\theta_A\sin\phi_A , \qquad (10)$$

$$v_C\sin\theta_C\cos\phi_C=v_B\sin\theta_B\cos\phi_B+v_A\sin\theta_A\cos\phi_A . \qquad (11)$$

In principle, the necessary partial derivatives may be found by expressing each "old" variable, $v_C, \cos\theta_C, \phi_C$ as an explicit function of the "new" variables, $v_A, \cos\theta_A, \phi_A$. However, as Eqs. (9)–(11) show, this procedure is quite clumsy. Instead, the following technique is used, in which Eqs. (9)–(11) give the old variables as implicit functions of the new ones. First, $\partial/\partial v_A$ operates on Eqs. (9)–(11). This yields three equations in the three unknowns, $\partial v_C/\partial v_A$, $\partial \cos\theta_C/\partial v_A$, and $\partial\phi_C/\partial v_A$. Similarly, when $\partial/\partial\cos\partial_A$ or $\partial/\partial\phi_A$ operates on Eqs. (9)–(11), we can solve the resulting set of three equations for $\partial v_C/\partial\cos\theta_A$, $\partial\cos\theta_C/\partial\cos\theta_A$, and $\partial\phi_C/\partial\cos\theta_A$, or $\partial v_C/\partial\phi_A$, $\partial\cos\theta_C/\partial\phi_A$, and $\partial\phi_C/\partial\phi_A$. In this manner all nine partial derivatives appearing in Eq. (4) are obtained.

We illustrate this procedure in part. Let $\partial/\partial\cos\theta_A$ operate on Eq. (10) giving

$$\frac{\partial v_C}{\partial\cos\theta_A}\sin\theta_C\sin\phi_C+\frac{\partial\sin\theta_C}{\partial\cos\theta_A}v_C\sin\phi_C + \frac{\partial\sin\phi_C}{\partial\cos\theta_A}v_C\sin\theta_C=v_A\sin\phi_A\frac{\partial\sin\theta_A}{\partial\cos\theta_A}. \qquad (12)$$

We use the chain rule to convert derivatives containing sine functions to those needed for the Jacobian. Thus:

$$\frac{\partial\sin\theta_C}{\partial\cos\theta_A}=\frac{\partial\sin\theta_C}{\partial\cos\theta_C}\frac{\partial\cos\theta_C}{\partial\cos\theta_A}=\frac{\partial\sqrt{1-\cos^2\theta_C}}{\partial\cos\theta_C}\frac{\partial\cos\theta_C}{\partial\cos\theta_A} =-\cot\theta_C\frac{\partial\cos\theta_C}{\partial\cos\theta_A}, \qquad (13)$$

and

$$\frac{\partial\sin\phi_C}{\partial\cos\theta_A}=\frac{\partial\sin\phi_C}{\partial\phi_C}\frac{\partial\phi_C}{\partial\cos\theta_A}=\cos\phi_C\frac{\partial\phi_C}{\partial\cos\theta_A}. \qquad (14)$$

Substituting (13) and (14) into (12) we obtain

$$\sin\theta_C\sin\phi_C\left(\frac{\partial v_C}{\partial\cos\theta_A}\right)-v_C\sin\phi_C\cot\theta_C\frac{\partial\cos\theta_C}{\partial\cos\theta_A} +v_C\sin\theta_C\cos\phi_C\left(\frac{\partial\phi_C}{\partial\cos\theta_A}\right)=-v_A\sin\phi_A\cot\theta_A . \qquad (15)$$

When all three sets of equations are solved for the desired partial derivatives, the results are

$$\frac{\partial v_C}{\partial v_A} = \cos\theta_A \cos\theta_C + \sin\theta_A \sin\theta_C \cos(\phi_A - \phi_C) , \tag{16}$$

$$\frac{\partial \cos\theta_C}{\partial v_A} = \frac{\sin\theta_C}{v_C} [\cos\theta_A \sin\theta_C - \cos\theta_C \sin\theta_A \cos(\phi_A - \phi_C)] , \tag{17}$$

$$\frac{\partial\phi_C}{\partial\phi_A} = \frac{\sin\theta_A \sin(\phi_A - \phi_C)}{v_C \sin\theta_C} , \tag{18}$$

$$\frac{\partial v_C}{\partial \cos\theta_A} = v_A [\cos\theta_C - \sin\theta_C \cot\theta_A \cos(\phi_A - \phi_C)] , \tag{19}$$

$$\frac{\partial \cos\theta_C}{\partial \cos\theta_A} = \frac{v_A}{v_C}[\sin^2\theta_C + \cos\theta_C \sin\theta_C \cot\theta_A \cos(\phi_A - \phi_C)] , \tag{20}$$

$$\frac{\partial\phi_C}{\partial \cos\theta_A} = \frac{v_A \cot_A \sin(\phi_C - \phi_A)}{v_C \sin\theta_C} , \tag{21}$$

$$\frac{\partial v_C}{\partial\phi_A} = v_A \sin\theta_A \sin\theta_C \sin(\phi_C - \phi_A) , \tag{22}$$

$$\frac{\partial \cos\theta_C}{\partial\phi_A} = -\frac{v_A}{v_C} \cos\theta_C \sin\theta_C \sin\theta_A \sin(\phi_C - \phi_A) , \tag{23}$$

$$\frac{\partial\phi_C}{\partial\phi_A} = \frac{v_A \sin\theta_A \cos(\phi_C - \phi_A)}{v_C \sin\theta_C} . \tag{24}$$

The evaluation of the determinant in Eq. (4) gives:

$$J\left(\frac{v_C, \Omega_C}{v_A, \Omega_A}\right) = \frac{v_A^2}{v_C^2} = \frac{T_A}{T_C} , \tag{25}$$

where T_A and T_C are the kinetic energies of the particle in the two frames.[2] For the special case where the exit channel velocity is constrained, such as in elastic scattering, Eq. (7) applies and we find that

$$J\left(\frac{\Omega_C}{\Omega_A}\right) = \frac{v_A^2}{v_C^2} [\cos\theta_A \cos\theta_C + \sin\theta_A \sin\theta_C \cos(\phi_A - \phi_C)] ,$$
$$= \frac{v_A^2}{v_C^2} \cos\delta , \tag{26}$$

where δ is the angle between $\mathbf{v}_A$ and $\mathbf{v}_C$.

The above results are general and apply to the three-dimensional transformation between two moving frames.[3] A special case of much interest is that in which the moving frame is the center-of-mass (c.m.) and the fixed frame is the laboratory reference frame (LAB).[4] Then Eqs. (25) and (26) become

$$J\left(\frac{v_C, \Omega_C}{v_A, \Omega_A}\right) = \frac{T_{\text{c.m.}}}{T_{\text{LAB}}} , \tag{27}$$

and

$$J\left(\frac{\Omega_C}{\Omega_A}\right) = \frac{T_{\text{c.m.}}}{T_{\text{LAB}}} \cos\delta . \tag{28}$$

Equation (27) agrees with the expression derived by Helbing,[5] Warnock and Bernstein,[6] and Wolfgang and Cross[7]; Eq. (28) agrees with the expression obtained by Morse and Bernstein.[8]

While these results are not new and seem to be well accepted in the field of crossed-beam molecular scattering, in the field of nuclear reactions, matters seem to be less clearly stated. Initially, we thought that there were clear instances of incorrect transformations in this literature,[9] but further consideration shows that we were confused by the omission of much detail in previously published accounts.

III. RELATIVISTIC TRANSFORMATION

This case differs from the nonrelativistic one in that we wish to transform momenta rather than velocities. Thus, the relativistic counterpart of Eqs. (9)–(11) are[10]

$$p_C \cos\theta_C = \gamma p_A \cos\theta_A + (\beta E_A / c) , \tag{29}$$

$$p_C \sin\theta_C \sin\phi_C = p_A \sin\theta_A \sin\phi_A , \tag{30}$$

$$p_C \sin\theta_C \cos\phi_C = p_A \sin\theta_A \cos\phi_A , \tag{31}$$

where $\beta = v_B/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. In the above $E_A = m_A[1-(v_A/v_C)^2]^{-1/2} c^2$ is the energy in the moving frame where m_A is the rest mass and c is the velocity of light.

We calculate the nine partial derivatives just as in the nonrelativistic case:

$$\frac{\partial p_C}{\partial p_A} = \gamma \cos\theta_A \cos\theta_C + \sin\theta_A \sin\theta_C \times \cos(\phi_C - \phi_A) + \gamma\beta \frac{p_{AC}}{E_A} \cos\theta_C , \tag{32}$$

$$\frac{\partial \cos\theta_C}{\partial p_A} = \frac{1}{p_C} \sin\theta_C \left\{ \gamma \cos\theta_A \sin\theta_C - \sin\theta_A \cos\theta_C \times \cos(\phi_C - \phi_A) + \gamma\beta \frac{p_{AC}}{E_A} \sin\theta_C \right\} , \tag{33}$$

$$\frac{\partial\phi_C}{\partial p_A} = \frac{1}{p_C} \frac{\sin\theta_A}{\sin\theta_C} \sin(\phi_A - \phi_C) , \tag{34}$$

$$\frac{\partial p_C}{\partial \cos\theta_A} = p_A [\gamma \cos\theta_C - \sin\theta_C \cot\theta_A \cos(\phi_C - \phi_A)] , \tag{35}$$

$$\frac{\partial \cos\theta_C}{\partial \cos\theta_A} = \frac{p_A}{p_C} \sin\theta_C [\gamma \sin\theta_C + \cos\theta_C \cot\theta_A \cos(\phi_C - \phi_A)] , \tag{36}$$

$$\frac{\partial\phi_C}{\partial \cos\theta_A} = \frac{p_A}{p_C} \frac{\cot\theta_A}{\sin\theta_C} \sin(\phi_C - \phi_A) , \tag{37}$$

$$\frac{\partial p_C}{\partial\phi_A} = p_A \sin\theta_A \sin\theta_C \sin(\phi_C - \phi_A) , \tag{38}$$

$$\frac{\partial \cos\theta_C}{\partial\phi_A} = -\frac{p_A}{p_C} \cos\theta_C \sin\theta_C \sin\theta_A \sin(\phi_C - \phi_A) , \tag{39}$$

$$\frac{\partial\phi_C}{\partial\phi_A} = \frac{p_A}{p_C} \frac{\sin\theta_A}{\sin\theta_C} \cos(\phi_C - \phi_A) . \tag{40}$$

These lead to the relativistic Jacobian

$$J\left(\frac{p_C, \Omega_C}{p_A, \Omega_A}\right) = \frac{p_A^2}{p_C^2} \frac{1}{E_A} \{\gamma E_A + \gamma\beta (p_A \cos\theta_A) c\} . \tag{41}$$

Since the energies transform according to

$$E_C = \gamma E_A + \gamma\beta (p_A \cos\theta_A) c , \tag{42}$$

the Jacobian for relativistic scattering reduces to

$$J\left(\frac{p_C, \Omega_C}{p_A, \Omega_A}\right) = \frac{p_A^2 E_C}{p_C^2 E_A} , \tag{43}$$

In the nonrelativistic limit, $(E_C/E_A) \to 1$, and we recover the Jacobian for nonrelativistic scattering given in Eq. (25). For elastic scattering the Jacobian of the relativistic transformation is

$$J\left(\frac{\Omega_C}{\Omega_A}\right)=\frac{p_A^2}{p_C^2}[\cos\theta_C\cos\theta_A+\gamma\sin\theta_C\sin\theta_A\cos(\phi_C-\phi_A)] . \quad (44)$$

In the nonrelativistic limit, γ approaches unity and Eq. (44) becomes identical to Eq. (26). Equations (43) and (44) agree with those derived previously by Hagedorn[11] and by Baldin, Goldanskii, and Rozenthal.[12]

In closing, it is worthwhile to point out that the differential operator technique employed in the above derivation is general and can be applied to calculate the transformation Jacobian for other variables and coordinate frames.

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APPENDIX: THE USE OF CONTINUOUS AND DISCRETE REPRESENTATIONS OF THE VELOCITY DISTRIBUTION IN TRANSFORMING SCATTERING DATA BETWEEN MOVING FRAMES

Let us suppose that a scattering process is characterized by a discrete number of exit channel velocities v_{Ai}, each associated with continuous angular variables Ω_A. We inquire under what conditions can we treat the transformation of scattering data between two moving frames as if the velocity distribution has a continuous spectrum.

We define a discrete probability function $P(v_{Ai}, \Omega_A)$ where v_{Ai} is fixed in value but Ω_A is not. The total cross section is given by

$$\sigma=\int_{\Omega_A}\sum_{i=1}^{n}P(v_{Ai}, \Omega_A)\,d\Omega_A , \quad (A1)$$

where n is the number of open exit channels.

To transform P from frame A to frame C we formally introduce the change of variables.

$$v_A=g(v_C, \Omega_C) , \quad (A2)$$

$$\Omega_A=h(v_C, \Omega_C) . \quad (A3)$$

For simplicity, we assume in what follows that all transformations are single valued. Then,

$$\begin{aligned}\sigma&=\int_{\Omega_C}\left(\frac{d\sigma}{d\Omega_C}\right)d\Omega_C ,\\ &=\int_{\Omega_C}J\left(\frac{\Omega_A}{\Omega_C}\right)\left(\frac{d\sigma}{d\Omega_A}\right)d\Omega_C ,\\ &=\int_{\Omega_C}\sum_{i=1}^{n}P[g(v_{Ci}, \Omega_C), h(v_{Ci}, \Omega)]\\ &\quad\times\frac{v_{Ci}^2\,d\Omega_C}{[g(v_{Ci}, \Omega_C)]^2\cos\delta} . \end{aligned}\quad (A4)$$

Let us suppose that the differential cross section is represented instead by a function $U(v_A, \Omega_A)$ that is a continuous function of v_A and Ω_A. Then the total cross section is derived from U according to

$$\sigma=\int_{\Omega_A}\int_{v_A}U(v_A, \Omega_A)\,dv_A\,d\Omega_A ,$$

and the transformation from frame A to frame C is accomplished by

$$\begin{aligned}\sigma&=\int_{\Omega_C}\int_{v_C}\left(\frac{d^2\sigma}{dv_C\,d\Omega_C}\right)dv_C\,d\Omega_C ,\\ &=\int_{\Omega_C}\int_{v_C}J\left(\frac{v_A, \Omega_A}{v_C, \Omega_C}\right)\left(\frac{d^2\sigma}{dv_A\,d\Omega_A}\right)dv_C\,d\Omega_C ,\\ &=\int_{\Omega_C}\int_{v_C}U[g(v_C, \Omega_C), h(v_C, \Omega_c)]\,\frac{v_C^2\,dv_C\,d\Omega_C}{[g(v_C, \Omega_C)]^2} . \end{aligned}\quad (A5)$$

In what follows we prove that whenever $U(v_A, \Omega_A)$ can be approximated by a finite sum of delta functions centered on v_{Ai} and weighted by $P(v_{Ai}, \Omega_A)$ i.e.,

$$U(v_A, \Omega_A)=\sum_{i=1}^{n}P(v_{Ai}, \Omega_A)\,\delta(v_A-v_{Ai}) , \quad (A6)$$

then (A5) becomes equivalent to (A4). Substituting (A6) into (A5) and making the appropriate change of variables:

$$\begin{aligned}\sigma&=\int_{\Omega_C}\int_{v_C}\sum_{i=1}^{n}P[g(v_{Ci}, \Omega_C), h(v_{Ci}, \Omega_C)]\\ &\quad\times\delta[g(v_C, \Omega_C)-g(v_{Ci}, \Omega_C)]\\ &\quad\times\frac{v_C^2}{[g(v_C, \Omega_C)]^2}\,dv_C\,d\Omega_C . \end{aligned}\quad (A7)$$

However, $\delta(x)=(\partial f/\partial x)\delta[f(x)]$, and with the help of Eq. (16) we see that

$$\begin{aligned}\delta[g(v_C, \Omega_C)-g(v_{Ci}, \Omega_C)]&=\frac{\delta(v_C-v_{Ci})}{\partial g/\partial v_C} ,\\ &=\frac{\delta(v_C-v_{Ci})}{\cos\delta} . \end{aligned}\quad (A8)$$

By substituting (A8) into (A7) and performing the integration over v_C one obtains an expression identical to (A4).

Let us return to Eq. (A5). It is possible to rewrite the integral over v_C as a sum of integrals over regions enclosing v_{Ci}:

$$\begin{aligned}&\int_{\Omega_C}\int_{v_C}U(g, h)\,(v_C^2/g^2)\,dv_C\,d\Omega_C\\ &\quad=\int_{\Omega_C}\sum_{i=1}^{n}\int_{v_{Ci}-\epsilon_i'}^{v_{Ci}+\epsilon_i}U(g, h)\,(v_C^2/g^2)\,dv\,d\Omega_C , \end{aligned}\quad (A9)$$

where the v_{Ci} are chosen to be those values of v_C corresponding to the v_{Ai} in Eq. (A6). When ϵ_i and ϵ_i' are properly chosen, the mean value theorem causes this equality to hold. If the resolution of the experiment is such that the v_{Ci} values begin to overlap, i.e., there are no gaps between ϵ_i's, then Eq. (A9) becomes

$$\int_{\Omega_C}\int_{v_C}U(g, h)\,(v_C^2/g^2)\,dv_C\,d\Omega_C\approx\int_{\Omega_C}\sum_{i=1}^{n}U[g(v_{Ci}, \Omega_C) , h(v_{Ci}, \Omega_C)]\,\frac{v_{Ci}^2\,\Delta v_{Ci}\,d\Omega_C}{[g(v_{Ci}, \Omega_C)]^2} . \quad (A10)$$

Realizing that $\Delta v_{Ci}/\Delta v_{Ai} \approx 1/\cos\delta$, and that Δv_{Ai} is of unit width in the A frame, then we recover the result that

$$\sigma = \int_{\Omega_C} \sum_{i=1}^{n} U[g(v_{Ci}, \Omega_C), h(v_{Ci}, \Omega_C)] \times \frac{v_{Ci}\,\Delta v_{Ai}\,d\Omega_C}{[g(v_{Ci}, \Omega_C)]^2 \cos\delta}, \quad \text{(A11)}$$

from which we identify that $P(v_{Ai}, \Omega_A) = U(v_A, \Omega_A)\Delta v_{Ai}$ by comparing Eq. (A11) to Eq. (A4). Thus in passing between the discrete and the continuous representations of velocity data, we associate $U(v_A, \Omega_A)$ with $P(v_{Ai}, \Omega_A)/\Delta v_{Ai}$ gives $P(v_{Ai}, \Omega_A)$ a "width." This so-called "width" is just the inherent dispersion in the scattering data caused in part by the lack of resolution of the detector.

[1]L. I. Schiff, *Quantum Mechanics*, 3rd ed. (McGraw-Hill, New York, 1968), pp. 110–114.

[2]Sometimes it is desired to relate $d^2\sigma/dT_A d\Omega_A$ to $d^2\sigma/dT_C d\Omega_C$; the Jacobian for this transformation is

$$J\left(\frac{T_A, \Omega_A}{T_C, \Omega_C}\right) = \left(\frac{T_A}{T_C}\right)^{1/2}.$$

[3]If the scattering process is restricted to a plane, the Jacobian of the transformation is a 2×2 determinant which yields

$$J\left(\frac{v_C, \theta_C}{v_A, \theta_A}\right) = \left(\frac{v_A}{v_C}\right).$$

[4]Had the three-dimensional general case been reduced to the evaluation of a 2×2 determinant by omitting partial derivatives involving the angle ϕ, the result is

$$J_{2\times 2}\left(\frac{v_C, \cos\theta_C}{v_A, \cos\theta_A}\right) = \left(\frac{v_A}{v_C}\right)\left(\frac{\sin\theta_C}{\sin\theta_A}\right),$$

where $J_{2\times 2}$ is recognized as the cofactor of $(\partial\phi_C/\partial\phi_A)$ in Eq. (5). This expression is similar to Eq. (25) when the latter is rewritten as

$$J\left(\frac{v_C, \Omega_C}{v_A, \Omega_A}\right) = \left(\frac{v_A}{v_C}\right)\left[\frac{\sin(\theta_B + \theta_C)}{\sin(\theta_A + \theta_B)}\right],$$

For $\theta_B = 0$ as in the case of the LAB to c.m. transformation, the above expression reduces to $J_{2\times 2}$. This identity arises from the special symmetry of the LAB to c.m. transformation which allows the incomplete solution of the Jacobian determinant to yield the correct result. This point is further clarified by examining Eq. (24). Only when $\phi_A = \phi_C$ and $\theta_B = 0$ does $(\partial\phi_C/\partial\phi_A) = 1$.

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